

Adiabatic Theory and Wave Action Density

→ Quasiparticle Picture

Wave Adiabatic Theory / Wave Kinetics

- Frequently encountered ^{continuum} problems with slowly varying parameters \Rightarrow adiabatic theory needed

\Rightarrow

- wave kinetic equation (consequence of Liouville Thm.)

$$\partial_t N + (\underline{v}_g + \underline{v}) \cdot \nabla N - \partial_x (\omega + \underline{k} \cdot \underline{v}) \cdot \partial_k N$$

= $\mathcal{L}(N)$; obvious analogy to Boltzmann Eqn.

$N \equiv \Sigma / \omega_k \equiv$ wave action density / wave energy density

\downarrow
wave energy density $\Sigma \equiv \frac{\partial}{\partial \omega} (\omega \epsilon_k) \Big|_{\omega_k} \frac{|E_{\underline{k}}|^2}{8\pi}$, for e.s. waves

$N \leftrightarrow \mathcal{E}$

Characteristics:

refraction by shear

$$\frac{dx}{dt} = \frac{\partial \omega}{\partial k} \hat{k} + \underline{v}, \quad \frac{dk}{dt} = - \frac{\partial}{\partial x} (\omega + \underline{k} \cdot \underline{v})$$

\downarrow
refraction by parametric variation

- need:

$$\omega \ll \frac{1}{\lambda} \frac{d\lambda}{dt}$$

$\lambda \equiv$ parameter

\Rightarrow space and time scale separation

$$\frac{1}{N} (\underline{v}_g \cdot \nabla N) \ll \omega \quad \Rightarrow \quad \underline{\Sigma} \cdot \underline{v}_g \ll \omega$$

Transport Egn = ρM

$$\frac{\partial n}{\partial t} + v_{gr} \cdot \nabla n - \frac{\partial \omega}{\partial x} \cdot \nabla_{\frac{1}{\hbar k}} n = C(n)$$

$$\frac{\partial n}{\partial t} + \frac{\partial \hbar \omega}{\partial \hbar k} \cdot \nabla n - \frac{\partial \hbar \omega}{\partial x} \cdot \nabla_{\frac{1}{\hbar k}} n = C(n)$$

$$\Rightarrow \left[\frac{\partial n}{\partial t} + \frac{\partial \epsilon}{\partial p} \cdot \nabla n - \frac{\partial \epsilon}{\partial x} \cdot \nabla_{\frac{1}{p}} n = C(n) \right]$$

used for:

$$\frac{1}{\epsilon} \frac{\partial \epsilon}{\partial x} < \frac{1}{\lambda_{DB}}$$

$$\lambda_{DB} = \hbar / p$$

(CN) → interactions with comparable scale.
ignore here.

Examples:

- linear theory of Langmuir turbulence
i.e. when will phonon grow?
- QL theory of Langmuir turbulence
i.e. determine evolution of plasmas
energy → net impact?!
- drift waves and sheared flow.
→ transport equations, super-fluids

$$N = \frac{\Sigma}{\Omega}$$

→ dynamics!

Fundamentals of wave kinetics

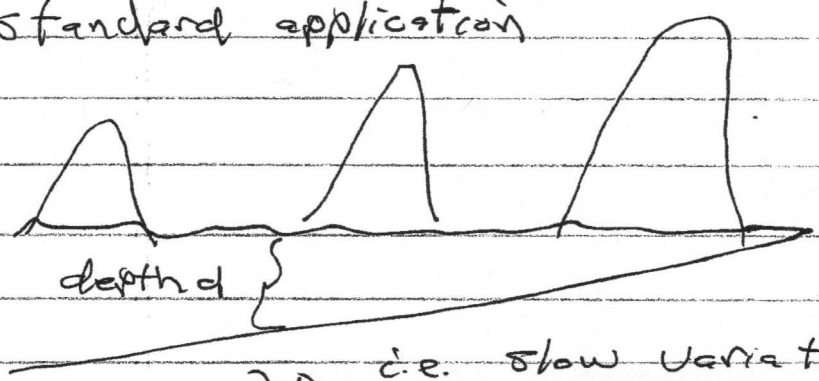
→ where does conservation of action emerge from?

→ answer:
Phase symmetry, underlies
of wave tract)
wave kinetics

→ approach via variational principle.

c.f. whitnam: "Linear and Nonlinear Waves"
Chapt. 14.

→ standard application



↪ i.e. slow variation

$$\frac{1}{d} \frac{d}{dx} d(x) \ll k$$

- influx of wave energy

- depth $H(x, y)$ decreases

⇒ wave amplification, breaking.

Derivation

Consider a system, [like cited ^{study} MHD] acoustics which can be described in terms of displacement $\underline{\xi}$; _{phase}

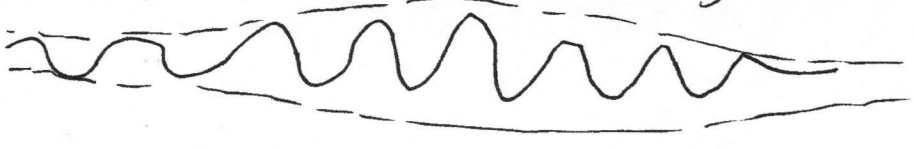
d.e. $\underline{\xi} = \text{re} \left\{ A e^{i\phi} + A^* e^{-i\phi} \right\}$

displacement can be viewed as excitation level

then wave equation arises from:

$$\delta S = \delta \int dt \int dx \mathcal{L}(\xi)$$

-Envision a wave train, with slowly varying amplitude, so eikonal approach optimal
d.e. fast variation in phase, aka WKB:



$$S = \int dt \int dx \mathcal{L}(\omega, \underline{k}, \underbrace{a}_{\text{amplitude}})$$

$$\begin{cases} \underline{k} = \underline{\nabla} \phi \\ \omega = -\dot{\phi} \end{cases}$$

$$= \int dt \int dx \mathcal{L}(-\dot{\phi}, \nabla \phi, a)$$

-neglect all corrections to eikonal theory. (no higher order WKB).

→ here L corresponds to period-averaged Lagrangian

- ϕ undetermined to const → phase symmetry!

∴ to vary:

$$\left. \begin{aligned} \delta S / \delta a &= 0 \\ \delta S / \delta \phi &= 0 \end{aligned} \right\} \Rightarrow \text{2 eqns}$$

Now, in linear theory:

$$[G(\omega, k) \Rightarrow G]$$

$$- \mathcal{L} = G(\omega, k) a^2$$

continuity

$$\begin{cases} G(\omega, k) = 0 & \text{disp} \\ \omega^2 = k^2 c_s^2 & \text{disp} \end{cases}$$

d.o for MHD, as in wave section:

$$\mathcal{L} = \frac{1}{2} \rho \dot{\underline{\Sigma}}^2 - \frac{1}{2} \rho \left[\underline{D}(k, \omega, t) \right]^2 \underline{\Sigma}^2$$

concrete form of Lagrangian

↳ eikonal form of stiffness matrix (→ potential energy)

$$\Rightarrow \underline{\Sigma} \cdot \underline{D} \cdot \underline{\Sigma}$$

$$\text{if: } \underline{\Sigma} = \underline{A} e^{i\phi} + \underline{A}^* e^{-i\phi}$$

↳ $\underline{D}(k, \omega, t)$, as for linear waves

$$\underline{\underline{L}} \quad \mathcal{G}(\omega, \underline{k}) = \frac{1}{2} \rho \left[\left(\frac{\partial \phi}{\partial t} \right)^2 - \left[\rho(\nabla \phi, \underline{x}, t) \right]^2 \right]$$

Now, 1) $\delta \mathcal{S} / \delta q = 0$

$$\Rightarrow \mathcal{G}(\omega, \underline{k}) = 0 \quad \rightarrow \text{dispn. relation}$$

but

$$\mathcal{G}(\omega, \underline{k}) = \rho \left(\frac{\partial \phi}{\partial t} \right)^2 - \left[\rho(\nabla \phi, \underline{x}, t) \right]^2$$

$$= \rho \omega^2 - \rho^2 \quad \rightarrow \text{stiffness factn.}$$

$$\Rightarrow \text{dispn. relation}$$

2) $\delta \mathcal{S} / \delta \phi = 0$

$$\delta \mathcal{S} = \int dt \int d^3x \left\{ \frac{\delta \mathcal{L}}{\delta(\dot{\phi}_t)} \delta(\dot{\phi}_t) + \frac{\delta \mathcal{L}}{\delta(\phi_{\underline{x}})} \delta(\phi_{\underline{x}}) \right\}$$

end pts fixed, c' b p

$$= \int dx \int d^3x \left\{ \partial_t \left(\frac{\delta \mathcal{L}}{\delta(\dot{\phi}_t)} \right) - \underline{\nabla} \cdot \left(\frac{\delta \mathcal{L}}{\delta(\phi_{\underline{x}})} \right) \right\} \delta \phi$$

$$\delta \mathcal{S} = 0 \Rightarrow$$

$$\partial_t \left(\frac{\delta \mathcal{L}}{\delta(\dot{\phi}_t)} \right) - \underline{\nabla} \cdot \left(\frac{\delta \mathcal{L}}{\delta(\phi_{\underline{x}})} \right) = 0 \quad \text{conservation eqn. } \downarrow$$

Now, have: $G(h, \omega) = 0$ (disph. reln.)

$$\partial_t \left(\frac{\partial G}{\partial \omega} \right) - \nabla \cdot \left(\frac{\partial G}{\partial \underline{h}} \right) = 0$$

$$dG = 0 \Rightarrow \frac{\partial G}{\partial \omega} d\omega + \frac{\partial G}{\partial \underline{h}} \cdot d\underline{h} = 0$$

$$\therefore v_{gr} = \frac{d\omega}{d\underline{h}} = \frac{-\partial G / \partial \underline{h}}{\partial G / \partial \omega} \quad (\text{akin } \omega)$$

$$\partial_t \left((\partial G / \partial \omega) a^2 \right) + \nabla \cdot \left[\frac{-\partial G / \partial \underline{h}}{\partial G / \partial \omega} \frac{\partial G}{\partial \omega} a^2 \right] = 0$$

and so $N \equiv \frac{\partial G}{\partial \omega} a^2$

$$\frac{\partial N}{\partial t} + \nabla \cdot (\underline{v}_{gr} N) = 0$$

though: $G = \rho \omega^2 - \rho^2$ (N not yet action)

$$\partial G / \partial \omega = 2\rho\omega = \frac{2\rho\omega^2}{\omega}$$

$$\frac{\partial G}{\partial \omega} a^2 \rightarrow \underline{\Sigma} / \omega$$

Also note energy is conserved \Leftrightarrow G invariant to time translations.

so, Noether's thm \Rightarrow there exists an ~~energy~~ energy conservation equation

have $\mathcal{L} = G(\underline{k}, \omega) a^2$

$$\partial \mathcal{L} / \partial a = 0 \Rightarrow G(\omega, \underline{k}) = 0$$

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \omega} \right) - \nabla \cdot \left(\frac{\partial \mathcal{L}}{\partial \underline{k}} \right) = 0$$

and of course!

$$\nabla \times \underline{k} = 0, \text{ as } \underline{k} = \nabla \phi$$

$$\frac{\partial \underline{k}}{\partial t} = -\frac{\partial \omega}{\partial \underline{x}}, \text{ as } \partial_t \nabla \phi = -\nabla \left(-\frac{\partial \phi}{\partial t} \right)$$

Now, $\mathcal{L} = 0$, as $G(\underline{k}, \omega) = 0$

as expect $\frac{\partial \mathcal{L}}{\partial \omega} \Rightarrow N$, $\omega \frac{\partial \mathcal{L}}{\partial \omega} \Rightarrow \mathcal{E}$
 $\nabla \cdot \left(\frac{\partial \mathcal{L}}{\partial \underline{k}} \right) \Rightarrow 0$, creatively \rightarrow dispersion relation satisfied

$$\frac{d}{dt} \left(\omega \frac{\partial \mathcal{L}}{\partial \omega} - \mathcal{L} \right) + \nabla \cdot \left[-\omega \frac{\partial \mathcal{L}}{\partial \underline{k}} \right] = 0$$

$\frac{-\partial G / \partial \underline{k}}{\partial G / \partial \omega} \frac{\partial G a^2}{\partial \omega}$
 $\nabla \cdot \underline{N} = \mathcal{E}$

$$\partial_t \left(\omega \underline{L} \dot{\underline{L}} \right) + \underline{D} \cdot \left(-\omega \frac{\partial \mathcal{L}}{\partial \dot{\underline{h}}} \right) = 0$$

check:

$$\begin{aligned} (\partial_t \omega) \underline{L} \dot{\underline{L}} + \omega \partial_t (\underline{L} \dot{\underline{L}}) - \underline{D} \cdot \left(-\omega \frac{\partial \mathcal{L}}{\partial \dot{\underline{h}}} \right) \\ + \underline{D} \cdot \left(-\omega \frac{\partial \mathcal{L}}{\partial \dot{\underline{h}}} \right) = 0 \end{aligned}$$

but $\partial_t \underline{L} \dot{\underline{L}} = \underline{D} \cdot (\underline{L} \dot{\underline{h}})$

$$\begin{aligned} (\underline{L} \dot{\underline{L}}) (\partial_t \omega) + \omega \underline{D} \cdot (\underline{L} \dot{\underline{h}}) - \omega (\underline{D} \cdot \underline{L} \dot{\underline{h}}) \\ - \left(\frac{\partial \mathcal{L}}{\partial \dot{\underline{h}}} \right) \cdot \underline{D} \omega - \frac{\partial \mathcal{L}}{\partial \dot{\underline{h}}} = 0 \end{aligned}$$

but $\partial_t \underline{h} = -\underline{D} \omega$ (Covariant derivative)

$$(\partial_t \omega) (\underline{L} \dot{\underline{L}}) + (\partial_t \underline{h}) \cdot \frac{\partial \mathcal{L}}{\partial \dot{\underline{h}}} - \frac{\partial \mathcal{L}}{\partial \dot{\underline{h}}} = 0$$

(identity) ✓

$$\partial_t \left\{ \omega \frac{\partial \mathcal{L}}{\partial \dot{\underline{h}}} \dot{\underline{L}} \right\} + \underline{D} \cdot \left(-\omega \frac{\partial \mathcal{L}}{\partial \dot{\underline{h}}} \right) = 0$$

But $G(\omega, k) = 0 \Rightarrow \mathcal{L} = 0$

\therefore

$$\partial_t \left\{ \omega \frac{\partial \mathcal{L}}{\partial \omega} \right\} + \nabla \cdot \left(-\omega \frac{\partial \mathcal{L}}{\partial \underline{k}} \right) = 0$$

Poynting Form

so

$$\Sigma \equiv \omega \frac{\partial \mathcal{L}}{\partial \omega} \rightarrow \begin{cases} \text{wave} \\ \text{energy density} \end{cases}$$

so

$$\frac{\partial \mathcal{L}}{\partial \omega} = \Sigma / \omega \rightarrow \begin{cases} \text{wave} \\ \text{action density} \end{cases}$$

$$= N(\underline{k}, \underline{x}, t) \rightarrow \text{adiabatic invariant for wave packet.}$$

so have:

$$\partial_t (N) + \nabla \cdot (\underline{v}_{gr} N) = 0$$

wave - kinetic

To demonstrate equivalence,

$$\frac{\partial N}{\partial t} + \underline{v}_{gr} \cdot \nabla N - \frac{\partial \omega}{\partial \underline{x}} \cdot \nabla_{\underline{k}} N = 0$$

and Liouville Thm:

$$\partial_t N + \nabla \cdot (\underline{v}_{gr} N) + \nabla_{\underline{k}} \cdot \left(-\frac{\partial \omega}{\partial \underline{x}} N \right) = 0$$

$\int d\underline{k}$, and assume narrow spread in \underline{k}
(i.e. wave packet) \Rightarrow

$$\frac{\partial N}{\partial t} + \nabla \cdot [\underline{v}_{gr} N] = 0$$

Observe:

\rightarrow \underline{v}_{class} -like equation in eikonal phase space $(\underline{x}, \underline{k})$

$$\frac{\partial N}{\partial t} + \underline{v}_{gr} \cdot \frac{\partial N}{\partial \underline{x}} - \frac{\partial \omega}{\partial \underline{x}} \cdot \frac{\partial N}{\partial \underline{k}} = 0$$

and

\rightarrow continuity-type equation in \underline{x} -space,
for packet

$$\frac{\partial N}{\partial t} + \nabla \cdot (\underline{v}_{gr} N) = 0$$

Also observe:

seemingly issue re:

$$\frac{\partial \underline{k}}{\partial t} = - \frac{\partial \omega}{\partial \underline{x}} \quad \text{vs} \quad \frac{d\underline{k}}{dt} = - \frac{\partial \omega}{\partial \underline{x}}$$

Now $\frac{\partial h}{\partial x} = -\frac{\partial \omega}{\partial x}$ is (Eulerian)
(partial) relation in x, t

$\frac{dh}{dt} = -\frac{\partial \omega}{\partial x}$ is (Lagrangian)
(total) relation, following
packet)
(here $\omega = \omega(h, x, t)$, as $G=0$)

$$\frac{dh}{dt} = \frac{\partial h}{\partial t} + \underline{v} \cdot \underline{\nabla} h$$

$$= -\frac{\partial \omega}{\partial x} + \frac{\partial \omega}{\partial h} \cdot \frac{\partial h}{\partial x}$$

$$\frac{\partial h}{\partial t} = -\frac{\partial \omega}{\partial x} \quad \text{agree!}$$

→ Now, can convert from N to E |

i.e. $N = E/\omega$

$$\frac{dN}{dt} \Big|_{\text{reyo}} = \frac{d}{dt} (E/\omega) = 0$$

$$\frac{1}{\omega} \frac{d\varepsilon}{dt} \Big|_{\text{rays}} - \frac{1}{\omega^2} \varepsilon \frac{d\omega}{dt} \Big|_{\text{rays}} = 0$$

$$\text{Now } \frac{d\omega}{dt} = \partial_t \omega + \frac{\partial \omega}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial \omega}{\partial y} \cdot \frac{dy}{dt}$$

From eikonal eqns:

$$= \partial_t \omega + \frac{\partial \omega}{\partial x} \cdot \frac{\partial \omega}{\partial y} - \frac{\partial \omega}{\partial y} \cdot \frac{\partial \omega}{\partial x}$$

$$\text{so } \frac{d}{dt} \partial_t \omega = 0 \quad \leftarrow \text{energy conserved.}$$

$$\therefore \frac{dN}{dt} = 0 \quad \Rightarrow \quad \frac{d\varepsilon}{dt} = 0$$

$$\text{so } \partial_t \varepsilon + \nabla_{gr} \cdot \underline{\nabla} \varepsilon - \frac{\partial \omega}{\partial x} \cdot \underline{\nabla}_y \varepsilon = 0$$

and exploiting Liouville's Thm, etc \Rightarrow

$$\boxed{\frac{d\varepsilon}{dt} = \partial_t \varepsilon + \nabla \cdot [\nabla_{gr} \varepsilon]} = 0$$

conserved
energy
density

So, for conservative case d.e. $\partial_t \omega = 0$

$$\partial_t \mathcal{E} + \nabla \cdot [U_{gr} \mathcal{E}] = 0$$

If stationary, $\partial_t \mathcal{E} = 0$

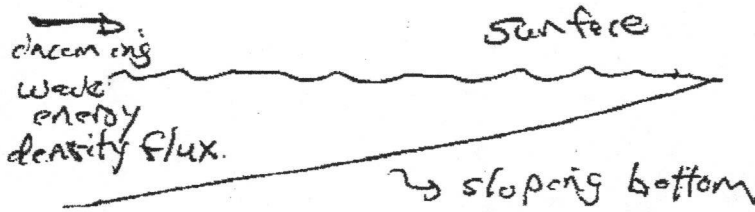
$$\Rightarrow \nabla \cdot [U_{gr} \mathcal{E}] = 0$$

incompressible
wave energy
flux ↓

⇒ U_{gr} drops ⇒
 $\mathcal{E} \uparrow \Rightarrow$ blocking,
breaking

(3) The beach...

Consider:



$$H = H(x)$$

Now, in shallow water ($\lambda > H$)



$$\frac{\partial h}{\partial t} + \frac{\partial (v h)}{\partial x} = 0$$

slope

shallow water eqns.

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = -g \frac{\partial h}{\partial x}$$

→ replaces pressure

$$v = v_0 + \tilde{v}, \quad h = H + \tilde{h}$$

$$\Rightarrow \begin{aligned} -c\omega \tilde{h} + ikH \tilde{v} &= 0 \\ -c\omega \tilde{v} &= -ckg \tilde{h} \end{aligned}$$

$$\therefore \omega^2 = k^2 g H \quad \text{is dispersion relation}$$

$\frac{\partial H}{\partial t} \leftrightarrow c_0^2$

→ analogy with acoustics is obvious

$$\begin{aligned} h &\leftrightarrow \psi & c_0^2 &= gH \\ v &\leftrightarrow v & & \text{etc.} \end{aligned}$$

energy \Rightarrow

15. ~~15.1~~

$$\frac{\partial \tilde{v}}{\partial t} = -g \frac{\partial \tilde{h}}{\partial x} \quad (1)$$

$$\frac{\partial \tilde{h}}{\partial t} = -H \frac{\partial \tilde{v}}{\partial x} \quad (2)$$

$$\Rightarrow (1) \times \tilde{v} + (2) \times \left(\tilde{v} \frac{\tilde{h}}{H} \right)$$

$$\therefore \frac{\partial \tilde{v}^2}{\partial t} = -g \tilde{v} \frac{\partial \tilde{h}}{\partial x}$$

$$\frac{g}{H} \frac{\partial \tilde{h}^2}{\partial t} = -\frac{g}{H} \tilde{h} \frac{\partial \tilde{v}}{\partial x}$$

$$\therefore \frac{\partial}{\partial t} \left(\frac{\tilde{v}^2}{2} + \frac{g \tilde{h}^2}{2H} \right) + \frac{\partial}{\partial x} \left(g \tilde{h} \tilde{v} \right) = 0$$

is energy theorem

$$\Rightarrow \Sigma = \frac{\tilde{v}^2}{2} + \frac{g \tilde{h}^2}{2H} \quad \text{is wave energy density.}$$

$$\omega/k = (gH)^{1/2} \quad \text{is wave phase velocity}$$

so ... so no explicit time dependence:

$$\frac{\partial \Sigma}{\partial t} + \nabla \cdot (v_{gr} \Sigma) = 0$$

$\Rightarrow v_g(x) \epsilon(x) = v_{g0} \epsilon_{00} = I$
 \downarrow
 incoming wave flux

$v_g = \sqrt{gH(x)}$
 \hookrightarrow shallow water waves have zero dispersion

$\therefore \sqrt{gH(x)} \epsilon(x) = I$

as $x \rightarrow$ shore $v_g \downarrow$ so wave energy ~~must~~ must increase.

Now $\epsilon(x) = \frac{\tilde{v}^2}{2} + \frac{g \tilde{h}^2}{2H} \approx \frac{g \tilde{h}^2}{2H}$

$\sqrt{gH(x)} \frac{g \tilde{h}^2}{2H(x)} = I$

$\frac{\tilde{h}^2}{H(x)^2} = \frac{2I}{(\sqrt{g})^3} (\sqrt{H(x)})^{-3}$

ch.e. $\left(\frac{\tilde{h}}{H}\right)^2 \sim (\text{const}) I / (H(x))^{3/2}$

e. $\tilde{h}/H \rightarrow 1 \iff$ breaking \iff as $H(x)$ drops.

N.B.:

→ if know bottom profile, can deduce displacement profile, and approximate reshing point.

→ 2D bottom contours \Rightarrow wave refraction

$$\frac{dk}{dt} = -\frac{\partial \omega}{\partial x} = -kg \left(\frac{\partial H(x,y)}{\partial x} \right)$$

d.e. wave fronts tend to align with bottom contours approaching shore.